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Note

Recognition of quasi-Meyniel graphs

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Abstract

We present a polynomial-time algorithm for recognizing quasi-Meyniel graphs. A *hole* is a chordless cycle with at least four vertices. A *cap* is a cycle with at least five vertices, with a single chord that forms a triangle with two edges of the cycle. A graph G is *quasi-Meyniel* if it contains no odd hole and for some $x \in V(G)$, the chord of every cap in G has x as an endvertex. Our recognition algorithm is based on star cutset decompositions. © 2001 Elsevier Science B.V. All rights reserved.

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1. Quasi-Meyniel graphs

A cycle is *even* if it contains an even number of vertices, and it is *odd* otherwise. A *hole* is a chordless cycle with at least four vertices. A *cap* is a cycle with at least five vertices, with a single chord that forms a triangle with two edges of the cycle. A graph G contains a graph H if H is an induced subgraph of G . A graph is *H-free* if it does not contain H .

A graph is *Meyniel* if every odd cycle with at least five vertices has two or more chords [11]. Clearly, a graph is Meyniel if and only if it contains no odd hole and no cap. A graph G is *quasi-Meyniel* if it contains no odd hole, and if it contains a *tip*: a vertex x such that the chord of every cap in G has x as an endvertex. This class of graphs is introduced by Hertz [9], where he gives an efficient coloring algorithm for Meyniel graphs, by coloring quasi-Meyniel graphs with given tips.

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Every Meyniel graph G is quasi-Meyniel, with every vertex of G being a tip. If G is quasi-Meyniel but not Meyniel, then G must contain a cap and G has at most two tips. If G is quasi-Meyniel with tip x , then every induced subgraph of G containing x is quasi-Meyniel with tip x and every induced subgraph of G not containing vertex x is Meyniel.

The goal of this paper is to address two algorithmic questions: how to find a tip of a quasi-Meyniel graph, and how to recognize quasi-Meyniel graphs.

Finding a tip of a quasi-Meyniel graph is important for coloring purposes. The search for polynomial-time algorithms for classes of graphs defined by chordality conditions has attracted much interest, especially in the context of Perfect Graph Theory [4–8,10]. A graph G is perfect if, for all induced subgraphs H of G , the size of the largest clique in H is equal to the chromatic number of H . A long-standing conjecture of Berge [1] states that G is perfect if and only if neither G nor its complement contain an odd hole. The existence of a polynomial-time algorithm to test whether G contains an odd hole implies a polynomial-time algorithm to test whether G is perfect, modulo the verification of Berge's conjecture, and it is possible that such an algorithm may itself prove the conjecture.

Finding a cap in a graph G or verifying that G does not contain one can be done as follows. For every edge xy in G and for every $z \in N(x) \cap N(y)$, check whether x and y both have a neighbor in the same component C of $G \setminus ((N(x) \cap N(y)) \cup N(z))$. If they do, then the node set $\{x, y, z\} \cup V(P)$, where P is a shortest path in C whose one endnode is adjacent to x and the other to y , induces a cap with chord xy . If the condition fails for all choices of nodes x, y and z , then G does not contain a cap.

Note that by finding chords of all of the caps of a graph G , one can easily find a tip candidate of G : a node that is an endnode of the chord of every cap. But this is not sufficient to recognize quasi-Meyniel graphs since one still has to check whether there is an odd hole that passes through the tip, and this problem is NP-complete in general [2].

The recognition of Meyniel graphs in polynomial time was established by Burlet and Fonlupt [3] when they defined the amalgam decomposition and proved that the amalgam of two Meyniel graphs is a Meyniel graph and, conversely, that any Meyniel graph can be amalgam decomposed in polynomial time into basic Meyniel graphs, which in turn can be recognized in polynomial time. The polynomial-time algorithm for recognizing quasi-Meyniel graphs that we propose here is based on a decomposition through star cutsets which preserves in both senses, ascending and descending, the property of being quasi-Meyniel. This decomposition for quasi-Meyniel graphs yields a decomposition tree whose leaves are Meyniel graphs.

2. Decomposition

Given a graph G and $S \subseteq V(G)$, we denote by $G \setminus S$ the subgraph of G induced by the vertex set $V(G) \setminus S$. A node set $S \subseteq V(G)$ is a *cutset* of a connected graph G if the graph $G \setminus S$ is disconnected. A node set $S \subseteq V(G)$ is a *star cutset with center x* of G if

S is a cutset of G and some $x \in S$ is adjacent to all the vertices of $S \setminus \{x\}$. Let a node set S be a cutset of a graph G , and let C_1, \dots, C_n be the connected components of $G \setminus S$. The *blocks of decomposition* by S are graphs G_1, \dots, G_n where G_i is the subgraph of G induced by the vertex set $V(C_i) \cup S$.

Lemma 1. *Suppose that a graph G and a vertex $x \in V(G)$ are such that $G \setminus \{x\}$ is Meyniel. Let S be a star cutset of G with center x , and let G_1, \dots, G_n be the blocks of decomposition by S . Then G is quasi-Meyniel with tip x if and only if G_i is quasi-Meyniel with tip x , for every i .*

Proof. If G is quasi-Meyniel with tip x , so are all the G_i 's, since they are induced subgraphs of G containing vertex x . To prove the converse, assume that G contains an odd hole H or a cap H in which x is not an endvertex of a chord. Since $G \setminus \{x\}$ is Meyniel, H contains x . But then $H \setminus S$ is contained in some connected component C_i of $G \setminus S$, and so H is contained in G_i . Hence, G_i is not quasi-Meyniel with tip x . \square

A *wheel*, denoted by (H, x) , is a graph induced by a hole H and a vertex $x \notin V(H)$ having at least three neighbors in H . Vertex x is the *center* of the wheel. A *sector* of the wheel is a subpath of H whose endvertices are neighbors of x and intermediate nodes are not. A *short sector* is a sector of length 1, and a *long sector* is a sector of length at least 2. A *twin wheel* is a wheel with three sectors, two of which are short.

Given a triangle $\{x_1, x_2, x_3\}$ and a vertex y adjacent to at most one vertex in $\{x_1, x_2, x_3\}$, a $3PC(x_1x_2x_3, y)$ is a graph induced by three chordless paths, $P_1 = x_1 \dots y$, $P_2 = x_2 \dots y$ and $P_3 = x_3 \dots y$, having no common vertices other than y , and such that the only adjacencies between the vertices of $P_1 \setminus \{y\}$, $P_2 \setminus \{y\}$, and $P_3 \setminus \{y\}$ are the edges of the triangle $\{x_1, x_2, x_3\}$. Note that $V(P_i) \cup V(P_j)$ induces a hole, when $i \neq j$.

The following simple facts will be used in the proof of Lemma 2 below:

Fact 1. Let (H, y) be a wheel in a quasi-Meyniel graph G with tip x , such that $y \neq x$.

If (H, y) has both a short and a long sector, then (H, y) is a twin wheel.

Fact 2. Odd-hole-free graphs cannot contain a $3PC(x_1x_2x_3, y)$.

Lemma 2. *Let G be a quasi-Meyniel graph with tip x . Suppose G contains a cap induced by a hole $H = xx_1 \dots x_kx$ and a vertex y adjacent to x and x_k . Then $S = (N(x) \cup \{x\}) \setminus \{y\}$ is a cutset separating y from $H \setminus S$.*

Proof. We prove that $T = (N(x) \cup \{x\}) \setminus \{y, x_1\}$ is a cutset separating y from $H \setminus T$, which clearly implies the lemma. Suppose not and let $P = p_1 \dots p_n$ be a path in $G \setminus T$ such that p_1 is adjacent to y , p_n is adjacent to a node of $H \setminus T$, and no proper subset of $V(P)$ induces a path with these properties. Note that x does not have a neighbor in P , but x_k possibly does. Let x_i be the vertex of H with lowest index adjacent to p_n . Note that $i < k$. Let H' be the hole induced by the vertex set $V(P) \cup \{y, x, x_1, \dots, x_i\}$. Let x_j be the neighbor of p_n in $H \setminus \{x_k\}$ with highest index. Note that possibly $i = j$. We now consider the following two cases.

Case 1: x_k is adjacent to a vertex of P . Then (H', x_k) is a wheel that contains at least one short sector (since x_k is adjacent to x and y) and at least one long sector (since x_k is not adjacent to x_1). Hence, by Fact 1, (H', x_k) is a twin wheel. So the only neighbor of x_k in P is p_1 . If $n \neq 1$ or $j \neq k-1$, then the vertex set $V(P) \cup \{y, x_j, \dots, x_k\}$ induces a cap, contradicting the assumption that x is a tip of G . Hence, $n = 1$ and $j = k-1$. If $i = j$, then (H', x_k) is a wheel that contradicts Fact 1. Hence, $i \neq j$, and so (H, p_1) is a wheel. By Fact 1, (H, p_1) is a twin wheel. So the neighbors of p_1 in H are x_k, x_{k-1} and x_{k-2} . But then the vertex set $(V(H) \cup \{y, p_1\}) \setminus \{x_k\}$ induces a cap, contradicting the assumption that x is a tip of G .

Case 2: x_k is not adjacent to a vertex of P . If $i = j$, then the vertex set $V(H) \cup V(P) \cup \{y\}$ induces a $3PC(xy, x_k, x_i)$, contradicting Fact 2. Hence $i \neq j$. If $x_i x_j$ is an edge, then the vertex set $V(P) \cup \{y, x_i, \dots, x_k\}$ induces a cap, contradicting the assumption that x is a tip of G . Otherwise, the vertex set $V(P) \cup \{x, x_1, \dots, x_i, x_j, \dots, x_k\} \cup \{y\}$ induces a $3PC(xy, x_k, p_n)$, contradicting Fact 2. \square

A *good star cutset with center x* is a cutset of the form $S = (N(x) \cup \{x\}) \setminus \{y\}$, where $y \in N(x)$ (the kind of a cutset used in Lemma 2).

3. Recognition algorithm

To test whether a graph G is quasi-Meyniel, we first test whether G contains a cap. If it does not, then it is sufficient to test whether G is Meyniel. If a cap with chord xy is detected, then we apply the recognition algorithm below twice, to check whether G is quasi-Meyniel with tip x and to check whether G is quasi-Meyniel with tip y .

Algorithm 1

Input: A graph G and a vertex $x \in V(G)$

Output: YES, if G is quasi-Meyniel with tip x , and NO otherwise

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if  $G \setminus \{x\}$  is not Meyniel then return NO
 $\mathcal{L}_1 \leftarrow G$ ,  $\mathcal{L}_2 \leftarrow \emptyset$ ;
while  $\mathcal{L}_1 \neq \emptyset$  do
  remove a graph  $F$  from  $\mathcal{L}_1$ 
  if there is a good star cutset  $S$  with center  $x$  in  $F$  then
    decompose  $F$  by  $S$  and add the blocks of decomposition to  $\mathcal{L}_1$ 
  else
    add  $F$  to  $\mathcal{L}_2$ 
if all the graphs in  $\mathcal{L}_2$  are Meyniel then
  return YES
else
  return NO

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Lemma 3. Algorithm 1 correctly identifies whether G is quasi-Meyniel with tip x .

Proof. If $G \setminus \{x\}$ is not Meyniel, then clearly G is not quasi-Meyniel with tip x , and Algorithm 1 correctly terminates. So, assume that $G \setminus \{x\}$ is Meyniel. Correctness of Algorithm 1 will follow from showing that G is quasi-Meyniel with tip x if and only if all the graphs in \mathcal{L}_2 are Meyniel. Algorithm 1 builds a decomposition tree whose internal nodes correspond to graphs having a good star cutset with center x , whereas the leaves correspond to graphs having no such cutset. Applying Lemma 1 to every graph corresponding to an internal node, we have that G is quasi-Meyniel with tip x if and only if all the graphs in \mathcal{L}_2 are quasi-Meyniel with tip x . Let $F \in \mathcal{L}_2$. We now show that F is quasi-Meyniel with tip x if and only if F is Meyniel. If F is Meyniel, then trivially F is quasi-Meyniel with tip x . To show the converse, assume that F is quasi-Meyniel with tip x , but that F is not Meyniel. Then F contains a cap with x being an endvertex of its chord. But then, by Lemma 2, F has a good star cutset with center x , contradicting the assumption that $F \in \mathcal{L}_2$. \square

3.1. Complexity analysis

Let G be a graph with n vertices and m edges. The above-described test for the existence of a cap takes time $\mathcal{O}(n^3m)$. The algorithm in [3] for checking whether a graph is Meyniel is $\mathcal{O}(n^7)$. More recently, Roussel and Rusu [12] showed how to do this recognition in time $\mathcal{O}(m^2 + mn)$. The complexity of the proposed quasi-Meyniel graph recognition is $\mathcal{O}(n^3m)$, being dominated by the complexity of Algorithm 1 which checks whether G is quasi-Meyniel with tip x , whose complexity we now establish to be $\mathcal{O}(mn^3 + m^2n + mn^2)$.

Consider an internal node of the decomposition tree and suppose it corresponds to decomposing the graph H with a good star cutset $S = (N(x) \cup \{x\}) \setminus \{y\}$. Let z be a node of a component of $H \setminus S$ that does not contain y . Label the corresponding internal node of the decomposition tree with pair (y, z) . Clearly no two internal nodes are labeled with the same pair, so the number of internal nodes in the decomposition tree is $\mathcal{O}(n^2)$. Decomposition through a good star cutset with center x can be performed in time $\mathcal{O}(mn)$: for every neighbor y of x , test whether $(N(x) \cup \{x\}) \setminus \{y\}$ is a cutset in time $\mathcal{O}(m)$. Thus the total cost of building the decomposition tree is $\mathcal{O}(mn^3)$.

The leaves in the decomposition tree that do not contain any non-neighbors of x are clearly Meyniel. The number of leaves that contain a non-neighbor of x is $\mathcal{O}(n)$, since no two distinct leaves can contain the same non-neighbor of x . So verifying whether the graphs in \mathcal{L}_2 are Meyniel can be implemented to run in time $\mathcal{O}(m^2n + mn^2)$, assuming an $\mathcal{O}(m^2 + mn)$ algorithm for testing whether a graph is Meyniel. Hence, Algorithm 1 can be implemented to run in time $\mathcal{O}(mn^3 + m^2n + mn^2)$.

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